Degrees of Truth: the formal logic of classical and quantum probabilities as well as fuzzy sets.

Logic is the study of reasoning. A language of propositions is fundamental to this study as well as true and false, the two primitive 'truth-values' assigned to propositions to express their meaning.

This paper is concerned with uncertain reasoning. Numbers including for example probabilities, can be used to express different 'levels' of uncertainty, here called degrees of truth. In this paper they will be derived from the primitives of traditional logic, from the two truth-values and from traditional laws of reasoning.

In section 1 truth-functional propositional logic $\mathcal{L}$ is developed from first principles and in section 2 the algebra of this logic is shown to be a slight generalisation of a Boolean algebra. Following sections show how this can be developed into a many-valued modal logic that expresses degrees of truth including classical and quantum probabilities as well as fuzzy sets. The Hilbert Space representation of quantum theories is discussed in section 8.

1. Formal logic $\mathcal{L}$

In formal logic mathematical methods are applied to the study of reasoning. Propositions of a language L are represented by variables, L = {p, q, r…}, and the truth-values true and false by symbols **t** and **f** respectively. *Valuations* are then introduced as functions assigning truth-values to propositions.

Defn 1.1: (valuation h, set H of $\mathcal{L}$)

> A (simple) *valuation* h of the logic $\mathcal{L}$ with language L, is a structure-preserving function from (simple) propositions to truth-values, h: L –> {**t, f**}. Each such h is in H, the set of all valuations of logic $\mathcal{L}$.

So simple valuations map simple propositions to the two truth-values in a way that respects their propositional relations. Generally no propositional relations over L are assumed in order to keep the discussion general, but these can be important as later discussion of mechanical theories shows.

Although every valuation h assigns truth-values to propositions, not *every* proposition in L must take a value. It is *not* assumed that the entire set L is the domain of every valuation h for several reasons. First this is not required by first principles, for a language expresses meaning when only some propositions have truth-values. Second, propositional relations on L may not *allow* bivalent assignments of the two truth-values, so assuming bivalence would prevent such languages being studied. And third, in a logic of uncertainties it is inappropriate to assume that every proposition is always true or false. Probabilities for example have particular interest in just those cases when truth-values are not assigned. For all these reasons truth-value "gaps" are allowed in logic $\mathcal{L}$.

Although only two truth-values are primitive to logic, *three* distinct relations between propositions and valuations can be derived from these and so there are three corresponding "logical values" in the truth-functional logic $\mathcal{L}$.

Defn 1.2: (valuation relations true, false and unassigned, logical values **t**, **f**, **u**)
For any h in H and p in L, there are three valuation relations: proposition p is *true in h* if h(p) = **t,** *p is false in h* if h(p) = **f,** or p is *unassigned by h* otherwise ie. when h(p) ≠ **t** and h(p) ≠ **f** in which case we write h(p) = **u.**

So proposition p may be true, false or unassigned in h, with corresponding *logical values* **t**, **f** and **u**. Although there are only two primitive truth-values a third logical value is derived to express the case where neither truth-value is assigned. It follows that each valuation h: L –> {**t**, **f**} corresponds to a *3-valued* mapping h: L –> {**t**, **f, u**} which has the whole of L in its domain. Since these two functions exactly correspond both will be called a valuation and will be represented by the same symbol h in H, without ambiguity.

Logical connectives allow simple propositions to be combined into complex ones. By definition the truth-value of a *truth-functional* connective in any valuation h is determined by the truth-value of its constituents in h. Traditional connectives are defined to satisfy the Laws of Thought. These include binary connectives $\wedge$, $\vee$, $\supset$, $\equiv$ for conjunction, disjunction, implication and equivalence respectively as well as a single unary connective $\neg$ for negation. The truth-functional logic $\mathcal{L}$ has the same traditional binary connectives, but having truth-value gaps allows *two* different unary truth-functional connectives for "not" to be distinguished, $\neg$ for negation and $\sim$ for denial.

Formally the *alphabet* of logic $\mathcal{L}$ includes the variables for simple propositions, L = {p, q, r,…}, the connective symbols $\wedge$, $\vee$, $\supset$, $\equiv$, $\neg$, $\sim$ as well as brackets. *Well-formed formulae* (wffs) of the logic $\mathcal{L}$ representing all simple or complex propositions are derived from these by the following rules.

Defn 1.3: (Rules of formation defining wffs of $\mathcal{L}$)
The well-formed formulae (wffs) of logic $\mathcal{L}$ are derived from propositional variables by Induction:
i) If p is a simple propositional variable, $p \in L$, then p is a wff of $\mathcal{L}$
ii) where α is a wff of $\mathcal{L}$ so too are $\neg\alpha$ and $\sim\alpha$
iii) where α, β are wffs of $\mathcal{L}$ so too are $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \supset \beta$, $\alpha \equiv \beta$
iv) normal conventions of bracketing apply

So any p in L representing a simple proposition of the logic $\mathcal{L}$ is a wff, and other wffs representing complex propositions are defined by induction from the traditional binary and two unary connectives. Variables α, β, γ,… are now assumed to range over the well-formed formulae of logic $\mathcal{L}$, and according to this definition $(\alpha \wedge (\beta \vee \gamma)$, $p \supset p$, and $\alpha \equiv (\beta \wedge \neg\gamma)$ for example are wffs representing complex propositions of $\mathcal{L}$.

The truth-functional connectives are defined by rules showing how any valuation of simple constituents determine the value of a complex wff in this valuation. These are conveniently expressed by logical or "truth" tables where each row corresponds to a different valuation.

Table 1.1: Valuation rules defining binary connectives in $\mathcal{L}$

α	β	i) $\alpha \wedge \beta$	ii) $\alpha \vee \beta$	iii) $\alpha \supset \beta$	iv) $\alpha \equiv \beta$
t	t	t	t	t	t
t	f	f	t	f	f
t	u	u	t	u	f
f	t	f	t	t	f
f	f	f	f	t	t
f	u	f	u	t	f
u	t	u	t	t	f
u	f	f	u	u	f
u	u	u	u	t	t

Table 2.2: Valuation rules defining unary connectives in $\mathcal{L}$

α	i) $\neg\alpha$	ii) $\sim\alpha$
t	f	f
f	t	t
u	u	t

These are the binary connectives of $\mathcal{L}_3$ the 3-valued logic of Lukasiewicz, with the two different available unary connectives. Table 2.2 i) defines negation $\neg$, in terms of opposite truth-values while ii) defines denial, $\sim$, in terms of failure to be true.

Rules defining connectives of $\mathcal{L}$ ensure as with traditional bivalent logic, that Boolean "Laws of Thought" are satisfied. The following formulae are *logically true* in $\mathcal{L}$, ie. they are true in every valuation of $\mathcal{L}$ as a construction of appropriate logical tables will show.

Defn: 1.4 (Boolean Laws of Thought)

Associative: $(\alpha \wedge (\beta \wedge \gamma) \equiv (\alpha \wedge \beta) \wedge \gamma$; $(\alpha \vee (\beta \vee \gamma) \equiv (\alpha \vee \beta) \vee \gamma$;
Commutative: $(\alpha \wedge \beta) \equiv (\beta \wedge \alpha)$; $(\alpha \vee \beta) \equiv (\beta \vee \alpha)$;
Distributive: $\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$; $\alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$
Identity: $\alpha \supset \alpha$
Double Negation: $\alpha \equiv \neg\neg\alpha$
Excluded Middle: $\alpha \vee \sim\alpha \equiv 1$

Satisfying the Commutative Law for example means that the order of constituents in a conjunction or disjunction does alter its valuation, which can be seen by Table 1.1 i) and ii), for example α is true and β false in a valuation the disjunction is true, and this is the same when β is true and α false, and likewise for the other cases. In a similar way one can create tables to show the other Laws are satisfied by the binary connectives. The two different Laws of Thought for "not" are traditionally supposed both to hold negation, which is the only unary connective in bivalent logic. However logic $\mathcal{L}$ has two different connectives and each is characterised by a different law, as the following table shows.

Table 1.3: Laws of Double Negation and Excluded Middle in $\mathcal{L}$

α	**i)**	$\neg$	$\neg$	α	$\equiv$	α	**ii)**	α	$\vee$	~	α
t		t	f	t	**t**	t		t	**t**	f	t
f		f	t	f	**t**	f		f	**t**	t	f
u		u	u	u	**t**	u		u	**t**	t	u

By table i) $\neg\neg\alpha$ has the same valuation as α in every valuation h, ie. the opposite of an opposite truth-value is the original one, $(\neg\neg\alpha \equiv \alpha)$ is logically true in $\mathcal{L}$ and the Law of Double Negation holds for this connective. By table ii) either a proposition or its denial must always be true, so $(\alpha \vee \sim\alpha)$ is logically true and the Law of Excluded Middle holds for this connective.

Since the truth-functional connectives in logic $\mathcal{L}$ defined by these tables above, satisfy the same Boolean Laws as the traditional bivalent propositional calculus, these two logics are essentially similar. One can check from the definitions above that they actually do coincide where only truth-values are assigned, ie. where only truth-values **t** and **f** are assigned then the definition of connectives is the same. In the next section algebraic methods will be used to further compare these two different logics.

Formally the valuation rules for connectives extend any simple valuation h of language L to a mapping from complex propositions as well, ie. any *simple* valuation h: L –> {**t**, **f**} is extended to, h: $\mathcal{L}$ –> {**t**, **f**}, and so the that the 3-valued map h: L –> {**t**, **f**, **u**} is also extended to h: $\mathcal{L}$ –> {**t**, **f**, **u**}. However since this extension is well-defined and uniquely determined for each simple valuation h, the distinction between simple and complex valuations is not maintained, with symbols h and H used for either.

Since logic $\mathcal{L}$ coincides with the traditional propositional calculus where bivalence is assumed, the 3-valued logic is more general than traditional bivalent logic. It is also more expressive since there are now two different unary operations for "not", making a distinction that is impossible to make when logic is bivalent, between a proposition being false and having no truth-value. The logic $\mathcal{L}$ is also capable of expressing its own valuation relations as the following connectives show.

Table 1.4: Connectives T, F, U expressing valuation relations:

α	i)	$T\alpha$	ii)	$F\alpha$	iii)	$U\alpha$
t		t		f		f
f		f		t		f
u		f		f		t

By table i) logic $\mathscr{L}$ can express the truth of a proposition for example by using connective T, which is true only when the constituent proposition is true and false otherwise. Similarly F and U express valuation relations false and unassigned respectively.

Further differences between logic $\mathscr{L}$ and the bivalent propositional calculus will now be discussed algebraically.

2. The algebra of logic $\mathscr{L}$

Algebra can simplify the representation of logical properties. For all propositions that are logically equivalent, ie. that share all valuation relations, are represented by a single element of an algebra. Logical connectives are then represented by operations among these elements, making their properties clear.

A logic based on traditional laws of reasoning, such as bivalent propositional logic or the 3-valued logic $\mathscr{L}$, are represented by a lattice. It is useful to review the definition of this structure that can be expressed in two different ways. First a lattice can be defined as a partially ordered set with the special property that any two elements have upper and lower bounds in the set. Secondly it can be defined as a particular kind of algebra.

Defn 2.1: (poset, lattice, $< A, \leq >$, $<A, \wedge, \vee >$)

i) A is a *partially ordered set* (poset) $<A, \leq>$, if relation $\leq$ is a partial ordering i.e. *reflexive*, $\mathbf{a} \leq \mathbf{a}$ for every $\mathbf{a}$ in A; *transitive*, if $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{b} \leq \mathbf{c}$ then $\mathbf{a} \leq \mathbf{c}$; and antisymmetric, if $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{b} \leq \mathbf{a}$ then $\mathbf{a} = \mathbf{b}$.

ii) A *lattice* is a partially ordered set in which every pair of elements a, b in A have a lattice meet $\wedge$ (least upper bound) and join $\vee$ (greatest lower bound) respectively in A[i]

iii) A *lattice* is an algebra $\mathscr{A} = <A, \wedge, \vee >$ where operations $\wedge, \vee$ are defined for any elements of A and satisfy the commutative, absorptive and associative laws. A partial ordering $\leq$ on $\mathscr{A}$ can be defined by setting $\mathbf{a} \leq \mathbf{b}$ iff $\mathbf{a} = (\mathbf{a} \wedge \mathbf{b})$ and $\mathbf{b} = (\mathbf{a} \vee \mathbf{b})$.

So a lattice has operations meet and join defined among all its elements and these satisfy the Laws of Thought for binary operations. A partial ordering relation, which has the important properties of set inclusion and hence deduction, can be defined.[ii]

Some special properties will be important.

Defn 2.2: (bounded, distributive lattice; Boolean algebra)

i) a *bounded* lattice has universal upper and lower bounds, 1 and 0 respectively, ie. $\mathbf{a} \leq 1$, $0 \leq \mathbf{a}$ for every $\mathbf{a}$ in A.

ii) a *distributive lattice* has meet and join that satisfy the distributive laws

iii) a *Boolean algebra* is a distributive lattice $< A, \wedge, \vee, ', 1, 0 >$ in which every element has an orthogonal-complement, ie. an operation ' where $\mathbf{a}'' = \mathbf{a}$ and $\mathbf{a} \vee \mathbf{a}' = 1$, $\mathbf{a} \wedge \mathbf{a}' = 0$.

So a *bounded* lattice has 1 and 0 elements that lie "above" and "below" all other elements respectively, with respect to the partial ordering. A distributive lattice satisfies the distributive identities and a Boolean algebra is a distributive lattice in which every element has an orthogonal-complement.

Propositional logic $\mathcal{L}$ is represented by its Lindenbaum-Tarksi algebra.[iii]

Defn 2.3: (algebra $\mathcal{A}$ of $\mathcal{L}$, equivalence classes $[\alpha]$)

The (representative or Lindenbaum-Tarski) algebra $\mathcal{A}$ of logic $\mathcal{L}$ is $\mathcal{A} = < A, \wedge, \vee, \perp, ' >$ where elements in A are *equivalence classes* of propositions $\mathbf{a} = [\alpha] = \{\gamma: \alpha \equiv \gamma$ is logically true in $\mathcal{L}\}$ and the operations represent logical connectives, ie. $\mathbf{a} \wedge \mathbf{b} = [\alpha \wedge \beta]$; $\mathbf{a} \vee \mathbf{b} = [\alpha \vee \beta]$; $\mathbf{a}\perp = [\neg\alpha]$; and $\mathbf{a}' = [\sim\alpha]$.

The operations on $\mathcal{A}$ are well-defined because they do not depend on the choice of proposition representing an equivalence class, a result which follows from the conditions defining logical connectives and from the fact that propositions in the classes are logically equivalent.[iv]

It is well-known that the algebra representing traditional 2-valued propositional calculus is a Boolean algebra, a result which follows because the connectives have been defined to satisfy the Boolean Laws.[v] The binary connectives of $\mathcal{L}$ are similarly defined and so they too are represented by operations of a distributive lattice. In this sense the deductive structure of logic $\mathcal{L}$ is entirely traditional.

However the two different unary connectives of logic $\mathcal{L}$ mean the algebra $\mathcal{A}$ of $\mathcal{L}$ is not a Boolean algebra but instead has two distinct unary operations.

Result 2.1: The algebra $\mathcal{A} = < A, \wedge, \vee, ', \perp >$ representing logic $\mathcal{L}$ is a distributive lattice with orthogonal and complement.

<u>Proof:</u> By discussion above binary operations are those of a distributive lattice. Operation $\perp$ representing negation is an orthogonal because since Double negation holds, $\neg\neg\alpha \equiv \alpha$ (by table above) and so $[\alpha]\perp\perp = [\neg\neg\alpha] =$

[α] and so by definition $\perp$ is an orthogonal on $\mathcal{A}$. Operation ' representing denial is a lattice complement because since Excluded Middle holds ($\alpha \vee \sim\alpha$) is logically true (by table above) and so $[\alpha] \vee [\alpha]' = [\alpha \vee \sim\alpha] = 1$ and by definition ' is a lattice complement and $\mathcal{A} = \langle A, \wedge, \vee, \leq, \perp, ' \rangle$ is a distributive lattice with orthogonal and complement.

Algebra $\mathcal{A}$ of logic $\mathcal{L}$ is thus an interesting generalisation of Boolean algebra in which all operations are Boolean, but instead of a single ortho-complement the algebra has separate orthogonal and complements. Algebra $\mathcal{A}$ is a distributive lattice with an orthogonal to represent negation and a lattice complement representing denial. The change from 2-valued to 3-valued logic is a generalisation that retains every traditional law.

In fact this algebra is *relatively Boolean* in an interesting sense.

Result 2.2: The algebra $\mathcal{A} = \langle A, \wedge, \vee, ', \perp \rangle$ representing logic $\mathcal{L}$ is *relatively Boolean* in the sense of being relatively ortho-complemented: any element **a** of $\mathcal{A}$ has an orthogonal-complement with respect to a subalgebra of $\mathcal{A}$.
Proof: Each element **a** in A determines a set C**a** the *logical context* or *scope* of **a** which is the set of equivalence classes containing propositions "above" ($\mathbf{a} \wedge \mathbf{a}\perp$) in $\mathcal{A}$ and "below" ($\mathbf{a} \vee \mathbf{a}\perp$), so $C\mathbf{a} = [C\alpha] = \{\gamma$: for some β in $\mathcal{L}$ both $(\alpha \wedge \neg\alpha) \supset \beta$ and $\beta \supset (\alpha \vee \neg\alpha)$ are logically true in $\mathcal{L}$ and so is $\gamma \equiv \beta\}$. It follows from properties of the logical connectives that C**a** is closed with respect to the operations and clearly by construction $\mathbf{a}\perp$ is not only an orthogonal but also a complement in this subsystem.

So each equivalence class of propositions determines a subsystem of elements with respect to which the class representing its negation has the traditional properties of a Boolean ortho-complement. This shows there is a natural sense of "logical relevance" or "context" among propositions, in the 3-vaued logic or equivalently a new sense of the "scope" of connectives, that simply does not occur in bivalent propositional calculus.

Defn 2.4**:** (Logical context or relevance)
For any wff α in $\mathcal{L}$ is $C\alpha = \{\beta$ in $\mathcal{L}$: $(\alpha \wedge \neg\alpha) \supset \beta$ logically true in $\mathcal{L}\}$. $C\alpha$ is the *logical context* of α, and any β in $C\alpha$ is *logically relevant* to α.

It follows from logical laws (De Morgan) that in this case $\beta \supset (\alpha \vee \neg\alpha)$ is also logically true, ie the propositions logically relevant to α are those logically implied by the conjunction of α and its negation and logically imply the disjunction of these propositions. Within this subsystem of propositions both senses of "not" will coincide and so this subsystem is entirely traditional.

Result 2.2 appears to be missed in discussions such as Rasiowa's of the algebraic representation of the Lukasiewicz logic. Yet this result shows clearly the close relationship between this logic $\mathcal{L}$ and the bivalent traditional 2-valued propositional calculus. The deductive structure of the two logics is the same since conjunction, disjunction and implication are in both cases represented by the operations and relations of a distributive lattice. "Not" is the same in both logics also in the sense that both the laws of Double Negation and of Excluded Middle are satisfied. The key difference between the two is that in a bivalent logic only one unary connective can be defined and so in this connectives both properties coincide and it is represented by an orthogonal-complement. In a 3-valued logic however the two properties of "not" are expressed by two separate connectives each satisfying a different law and so these are represented by an orthogonal and a complement. Result 2.2 shows that even in this case the properties of negation and denial coincide for each proposition among a subsystem of "logically relevant" propositions.

The 2-valued propositional calculus can now be understood as the special case of logic $\mathcal{L}$ where it is assumed that all propositions are mutually relevant, where the logical scope of every proposition is assumed to be the entire set of propositions. In the next section the logic $\mathcal{L}$ generates an even more interesting global logic that can express numerical and contextual modalities.

3. Global logic $\mathcal{L}$

Logic $\mathcal{L}$ is more expressive than the traditional bivalent propositional calculus, since the truth-value gaps allow two distinct senses of "not" to be expressed as well as valuation relations and logical "relevance" or "scope". Even with this change however the truth-functional connectives cannot express properties of uncertainties like probabilities or fuzzy sets. These depend on "global" features of the logic, on properties of multiple valuations that simply cannot be expressed truth-functionally.

Modal connectives introduced by Saul Kripke in the 1960's, are not truth-functional. These were defined by introducing new structures into logic, including a set of "possible worlds" with an "accessibility" relation over this set. Connectives *possible* and *certain* are defined in terms of these new structures, which are simply assumed to be primitive to logic. Different accessibility relations among the "worlds" generate different Kripke systems.

Here a different approach is taken. No new structures are added to the primitives of logic but instead new connectives are defined in terms of multiple valuations in H, in terms of subsets of these valuations generated by relations over H. The global logic $\mathcal{L}$ can include a wide range of different connectives generated by different relevance relations. This logic is more general than Kripke logics since it is an extension of the 3-valued propositional logic while the Kripke systems are extensions of the bivalent propositional calculus. Also the global conditions can generate Kripke "possible worlds" as a special case of the global connectives, which in addition can include different modalities based on different relations, while a Kripke logic is based on just one "accessibility" relation.[vi]

Since the global extension of logic $\mathcal{L}$ is generated only from its own set H of valuations, and relations over this set, there is no ambiguity if this global logic is called $\mathcal{L}$ as well. The simplest modalities of $\mathcal{L}$ depend on the entire set H. Connectives **L** and **M** are added to the alphabet of logic of $\mathcal{L}$ along with rules of formation that for any α in the propositional logic $\mathcal{L}$, **L**α and **M**α are wffs of its global logic. These connectives are then defined by global valuation rules.

Defn 3.1: (Simply certain and possible, **L M**)

For any α in propositional logic $\mathcal{L}$, any h in H,

i) Proposition α is (simply) *certain* according to valuation h, h(**L**α) = **t**, iff α is true in *all* valuations in H and h(**L**α) = **f** otherwise.

ii) **M** $=_{\mathrm{df}} \sim$ **L**$\sim \alpha$ so α is (simply) *possible* according to h, h(**M**α) = **t** iff α is true in *some valuation h'* in H, and h(**M**α) = **f** otherwise.

These valuation rules extend any valuation h of truth-functionall logic $\mathcal{L}$ to a valuation of these two global connectives as well. Since this extension is unique and well-defined for any h both the propositional and the global valuation is referred to by the same variable. Other connectives can obviously be derived from these two simple ones, for example α might be defined as *uncertain* in h when h(**L**α) ≠ **t** ie. when not every valuation in H finds it true.

More interesting uncertainties however are *contextual*, depending on a subset of valuations in H derived from some initial valuation h and a relation R over H. *Contextual* modalities depend on a proposition being true in *every* valuation related by R to h, or on some valuation in this subset. Elaborating the simple case above:

Defn 3.2: (Contextual modalities, $\mathbf{L}^R$, $\mathbf{M}^R$)

For any α in logic $\mathcal{L}$, h in H and R a binary relation over H then

i) Proposition α is *R-certain in valuation h*, h($\mathbf{L}^R\alpha$) = **t**, iff h'(α) = t for all h' such that h'Rh and h($\mathbf{L}^R\alpha$) = **f**, α is *R-uncertain in valuation h* otherwise.

ii) $\mathbf{M}^R =_{\mathrm{df}} \sim \mathbf{L}^R\sim$ and so α is *R-possible in valuation h,* h($\mathbf{M}^R\alpha$) = **t** iff h'(α) = **t** for some h' such that h' R h and h($\mathbf{M}^R\alpha$) = **f** otherwise.

A proposition is *R-certain in h* when if is true in all valuations related by R to h, and is *R-possible according to h* when true in *some* R-related valuation h'.

Different relations over H will obviously generate different modal connectives and unlike the Kripke logics, these can all be expressed in $\mathcal{L}$. For example a wff of form $\mathbf{L}^R\alpha \wedge \mathbf{M}^{R'}\beta$ is well-defined for distinct relations R, R' over H. Later some examples will be developed using different relations including consistency generating probabilities, and other relations defined by semantic conditions, for example by the propositions they find true.

The following terms are useful.

Defn 3.3: (truth-set T_h, falsity-set F_h, context C_h)

For any valuation h in H of logic $\mathcal{L}$, α a proposition of $\mathcal{L}$

i). The *truth-set of h* $T_h = \{\alpha \in \mathcal{L}: h(\alpha) = \mathbf{t}\}$

ii) The *falsity-set of h* $F_h = \{\alpha \in \mathcal{L}: h(\alpha) = \mathbf{f}\}$.

iii) The *context of h* $C_h = T_h \cup F_h$

So the *truth-set* of a valuation is the set of propositions it finds true, while the *falsity-set* contains the propositions false in this valuation. The *context* is the union of these two sets, containing every proposition assigned a truth-value.

Result 3.1: Truth sets and valuations exactly correspond.

<u>Proof</u>: Where valuation h = h' then $T_h = T_{h'}$ by defn h. Where $T_h = T_{h'}$ then $F_h = F_{h'}$ since these are negations of all propositions in T_h and so unassigned propositions are also the same (the propositions outside the context C_h) and so truth-sets and valuations exactly correspond.

So valuations are represented by their truth-sets, a fact that will be useful in using algebra to study global logic.

The difference between traditional propositional logic, propositional logic $\mathcal{L}$ and its global logic can now be clearly expressed. The traditional *2-valued propositional calculus* is the logic of contexts, appropriate where reasoning assumes every proposition has a truth-value. Truth-functional *logic* $\mathcal{L}$ having truth-value gaps is more general than bivalent logic, and is the logic of a single valuation h no longer limited to its context. Here the valuation relations of h can be expressed and reasoning includes propositions without a truth-value. Lastly the *global* logic $\mathcal{L}$ is yet more general since all truth-functional connectives are expressed as well as wider aspects of reasoning involving all the valuations. In global logic $\mathcal{L}$ one valuation can express aspects of other valuations as well.

The notion of a truth set and context allows global relations of mutual *relevance* and of *incompatibility* to be defined that could not be expressed in the truth-functional logic.

Defn 3.4: (relevant valuations, incompatible propositions)

For h, h' in H of α, β in $\mathcal{L}$,

i) Valuations h, h' are *mutually relevant*, h' Rel h iff $Ch \cap Ch' \neq \emptyset$

ii) Two propositions α, β are *compatible,* α C β iff for some h $\alpha \in C_h$, and $\beta \in C_h$ and are *incompatible* otherwise

Two valuations are *relevant* when there is some proposition assigned a truth-value by them both. Two propositions are *compatible* when they are both assigned truth-values by the same valuation and *incompatible* when this is not the case.

An important correspondence between propositions and valuations will be useful in later discussions of uncertainties.

Defn 3.5: (characteristic wff α_h, characteristic valuation h_α)

i). The *characteristic proposition* α_h of valuation h is logically equivalent to the conjunction of all propositions h finds true, ie. for any valuation h in H, $\alpha_h = \beta$ where $(\beta \equiv (\wedge \gamma_i))$ is logically true in $\mathcal{L}$ for all propositions γ_i in the truth-set T_h

ii). The *characteristic valuation* h_α of proposition α is the valuation finding this proposition and all its logical consequences true, but making no other truth-assignments, ie. for any wff α of logic $\mathcal{L}$, h_α in H is such that $T_{h_\alpha} = \{\beta: \alpha \supset \beta$ is logically true in $\mathcal{L}\}$.

So any valuation h in H generates a *characteristic proposition* α_h equivalent to the conjunction of all the propositions true in h. Similarly any proposition α generates a *characteristic valuation* h_α which assigns true to every logical consequence of α. Clearly where two valuations are the same, so too are their characteristic propositions, and vice versa two propositions that are logically equivalent have the same characteristic valuation.[vii]

Using this natural correspondence, uncertainties can be expressed in terms of valuations or propositions, something that will be particularly useful when probabilities are discussed.

4. Truth-systems of $\mathcal{L}$

The modalities *certain* and *possible* introduced above will not be further explored but instead will be elaborated into new numerical uncertainties that can very accurately express a sense of how certain or how possible a proposition is. These *degrees of truth* can be developed into probabilities and fuzzy set memberships. Intuitively they are refinements of the modalities that express not just whether *all* or *some* related valuations find a proposition true, but *how many* related valuations do so.

Mathematical measures are used for this refinement, which requires further algebra. For measures are functions which are well-defined only over a Boolean field of sets. The formal definitions are as follows.

Defn 4.1: (measure space, measure μ)

A *measure space* is a triple $< X, \mathcal{F}, \mu >$ where X is some set, $\mathcal{F}$ a Boolean field of subsets of X, and μ a function over $\mathcal{F}$ which has the following properties:

i) for any S in $\mathcal{F}$ $\mu(S) \geq 0$ and $\mu(S) \geq 1$

ii) $\mu(\varnothing) = 0$; $\mu(X) = 1$

iii) For disjoint sets S_i in $\mathcal{F}$, $\mu(\cup S_i) = \Sigma\, \mu(S_i)$

So a measure is a function operating over a Boolean field of sets that assigns a number in the interval [0, 1] to each subset. The properties are designed to express a sense of "size" of the set. Thus 0 is assigned to the empty set and 1 to the universal set X. The functions are additive so that the measure of a union of sets is the sum of the measures of individual sets where these are disjoint. It also follows that the measures are monotone, where $S \subset S'$ then $\mu(S) \leq \mu(S')$. The properties of the Boolean field of subsets over which a measure is defined obviously provide structure to the measures themselves, and so this Boolean structure is essential to the very notion of the "size" of a set. For this reason suggestions that measure theory might be modified to suit a non-Boolean algebra is not accepted, and instead finding appropriate measure spaces in the non-Boolean algebra of generalised logic $\mathscr{L}$ is considered the key to developing the numerical degrees of truth.

Traditionally the logical foundation for probability theory finds the measure space of a logic in the Stone Space of the Boolean algebra representing traditional bivalent propositional calculus. This close relation between Boolean algebra and bivalent logic has been largely unchallenged, perhaps in part because there is widespread reluctance to lose this correspondence: after all traditional bivalent logic is represented by a Boolean algebra that by Stone's Theorem is isomorphic to its Stone Space, a Boolean field of sets.[viii] This is the field of Boolean ultrafilters of the algebra of the logic, an "event space" for its probability theory, understood as representing the 2-valued valuations of the logic in which each proposition is true. The lack of such a close connection in any non-bivalent propositional may seem daunting.

However logic $\mathscr{L}$ being not bivalent has an algebra $\mathscr{A}$ that is not a Boolean algebra. There is now no corresponding Stone Space, no field of Boolean ultrafilters for this logic. It might appear that logic $\mathscr{L}$ therefore lacks a field of sets to act as a measure space, and it is sometimes suggested that the notion of a measure should be generalised to apply over structures that are not Boolean fields of sets, just as it is sometimes suggested that one of Boole's Laws of Thought be dropped.[ix] These suggestions have been neither fruitful nor beautiful. It hardly helps our understanding to lose a law of thought.

Boolean fields remain of central importance to the measures that define numerical degrees of truth, but these will represent "truth systems" of the logic, systems of valuations that find each proposition true. Some definitions are needed this precise. First are some properties of H.

Defn 4.2: (inclusion, agreement; maximal valuation)

Let h, h' be valuations in H with corresponding truth-sets T_h and $T_{h'}$

i) A partial ordering among valuations is generated by set inclusion among truth-sets: $h \subseteq h'$, (h' includes h or h' agrees with h) iff $T_h \subseteq T_{h'}$.

ii) Valuation h is *maximal* if its truth-set T_h is maximal with respect to set inclusion, ie. if there is no other valuation h' in H such that $T_h \subset T_{h'}$

So one valuation *includes* another (or *agrees* with it) if its truth-set includes the other's. A valuation is *maximal* when its truth-set is, ie. when no other truth-set properly contains its truth-set, ie. when no other valuation makes the same truth-assignments as a maximal valuation but also makes more.

Filters are subsets of an algebra with special properties, according to these standard definitions.

Defn 4.3: (filter, maximal filter, Boolean ultrafilter)

Let $\mathcal{A}$ = < A, ≤ > be a lattice.

i) Subset F of A is a *filter* of $\mathcal{A}$ if 1) $F \subset A$, 2) $F \neq \emptyset$ and 3) if $\mathbf{a} \wedge \mathbf{b} \in F$ then $\mathbf{a} \in F$ and $\mathbf{b} \in F$, and 4) if $\mathbf{a} \in F$ and $\mathbf{a} \leq \mathbf{c}$ then $\mathbf{c} \in F$.

ii) A *maximal filter* F is maximal with respect to set inclusion, ie. there is no other filter F' that properly contains it, no filter F' of $\mathcal{A}$ where $F \subset F'$.

iii) A *Boolean ultrafilter* F is a maximal filter of Boolean algebra $\mathcal{B}$ which has the property that for each element **a** in $\mathcal{B}$, either **a** or its orthgogonal-complement **a'** is contained in F, ie. $\mathbf{a} \in F$ or $\mathbf{a'} \in F$ for every **a**. This condition is sometimes called the *ultrafilter property*.

It will now be shown that filters of the algebra $\mathcal{A}$ of $\mathcal{L}$ represent truth-sets and hence also their valuations.

Result 4.1: Valuations in H, their truth-sets and the filters of $\mathcal{A}$ all exactly correspond.

Proof: By Result 3.1 above valuations and truth-sets exactly correspond. To show truth-sets and filters correspond let $[T_h]$ represent T_h on $\mathcal{A}$, ie. $[T_h] = \{[\alpha]: \alpha \in T_h\}$. To show this is a filter of $\mathcal{A}$ note first that T_h is *non-empty* since it contains at least the logical truths of $\mathcal{L}$, ie. $(\alpha \supset \alpha) \in T_h$ and so $[T_h] \neq \emptyset$ and condition 1) holds. This is a *proper* subset because it does not contain the negations of these truths, ie. $\neg(\alpha \supset \alpha) \notin T_h$ and so $[T_h] \subset A$ and condition 2) holds. Also $\mathbf{a} \wedge \mathbf{b} \in [T_h]$ iff $h(\alpha \wedge \beta) = \mathbf{t}$ by defn T_h, iff $h(\alpha) = \mathbf{t}$ and $h(\beta) = \mathbf{t}$ by defn $\wedge$, iff $\mathbf{a} \in [T_h]$ and $\mathbf{b} \in [T_h]$ defn $[T_h]$ so condition 3) holds. Similarly when $h(\alpha \supset \beta) = \mathbf{t}$ and $h(\alpha) = \mathbf{t}$ then $h(\beta) = \mathbf{t}$ as well by defn $\supset$, and so condition 4) holds and each T_h is represented by a filter $[T_h]$. To show vice versa, that any filter F of $\mathcal{A}$ represents a truth-set T_h for some h in H suppose it does not. Then either F contains inconsistent classes, fails to contain classes of logical consequences or fails to contain classes of conjuncts of propositions represented in F. In the first case for some α in $\mathcal{L}$, $[\alpha]$ and either $[\neg\alpha]$ or $[\sim\alpha]$ is in F, so $[\alpha \wedge \neg\alpha] = [\alpha \wedge \sim\alpha] = 0$ is in F by i) but this means every $[\beta]$ is in F because $0 \supset \beta$ is logically true in logic $\mathcal{L}$ and so $0 \leq [\beta]$ on $\mathcal{A}$ by ii) and so for any $\mathbf{b} = [\beta]$, $\mathbf{b} \in F$, which contradicts the assumption that F is a proper subset of $\mathcal{A}$ and so the assumption that F contains a contradiction must be false. The other cases are established in a similar way and it follows every filter $F = [T_h]$ for some h in H.

So filters, truth-sets and valuations all exactly correspond. This means that measuring sets of valuations that find a proposition true will be achieved by measuring sets of truth-sets containing the proposition, or by measuring sets of filters of $\mathcal{A}$ containing this proposition's equivalence class.

At last the Boolean fields appropriate as measure spaces for numerical uncertainties can be identified.

Defn 4.4: (truth-system $\mathcal{S}^{Rh}$, initial h, trivial h_0, simple truth-system $\mathcal{S}$)

For α a wff of logic $\mathcal{L}$, h in H, and R a binary relation over H

i) The *Rh-truth-system* of $\mathcal{L}$, $\mathcal{S}^{Rh} = < S^{\mathbf{Rh}}_{\boldsymbol{\alpha}}, \cap, \cup, \subset, - >$, where $S^{\mathbf{Rh}}_{\boldsymbol{\alpha}} = \{[T_{h'}]: h' \; R \; h \; \& \; h'(\alpha) = \mathbf{t}\}$ and the operations represent connectives, so $S^{\mathbf{Rh}}_{\boldsymbol{\alpha}} \cap S^{\mathbf{Rh}}_{\boldsymbol{\beta}} = S^{\mathbf{Rh}}_{\boldsymbol{\alpha \wedge \beta}}$, and $S^{\mathbf{Rh}}_{\boldsymbol{\alpha}} \cup S^{\mathbf{Rh}}_{\boldsymbol{\beta}} = S^{\mathbf{Rh}}_{\boldsymbol{\alpha \vee \beta}}$, and $- S^{\mathbf{Rh}}_{\boldsymbol{\alpha}} = S^{\mathbf{Rh}}_{\boldsymbol{\sim\alpha}}$.

ii) Valuation h is called the *initial condition* of truth-system $\mathcal{S}^{Rh}$.

iii) The truth-set T_0 of *trivial* valuation $h_0 = \{\alpha: h(\alpha) = \mathbf{t}$ for all h in H$\}$

iv) The *simple truth-system* $\mathcal{S}$ *of* $\mathcal{L}$ is the special case of $\mathcal{S}^{Rh}$ where $h = h_0$ and R is the universal relation over H, ie. for any h, h' in H, h R h'

So *truth-systems* of logic $\mathcal{L}$ are systems of sets of related filters of the algebra $\mathcal{A}$ of $\mathcal{L}$ containing each equivalence class. By Result 4.1 these represent systems of related truth-sets containing each proposition, and hence systems of valuations related by R to an *initial condition* h, that find each proposition true. Since the *trivial* valuation finds only logically true propositions true, and the *universal* relation relates all valuations, the simple truth-system $\mathcal{S}$ is the truth-system derived from all the valuations in H of $\mathcal{L}$.

Operations on truth-systems represent algebraic operations on $\mathcal{A}$ and thus the logical connectives of $\mathcal{L}$. In a fundamental result these are shown to be the operations of a Boolean algebra.

Result 4.2: Each truth-system $\mathcal{S}^{Rh}$ of logic $\mathcal{L}$ is a Boolean field of sets.

Proof follows from properties of logical connectives, which generate the operations on $\mathcal{S}^{Rh}$. It follows from properties of the binary logical connectives on $\mathcal{L}$ that set operations over $\mathcal{S}^{Rh}$ have the properties of a distributive lattice. To see it is a Boolean field note that *denial* generates the set complement operation on these systems. This connective obviously generates lattice complement (because it satisfies Excluded Middle) but in addition generates an involution (orthogonal) by this argument:

$S^{Rh}_{\sim\alpha} = \{[T_{h'}]: h' \; R \; h \; \& \; h'(\sim\alpha) = \mathbf{t}\}$ (by the definition of S^{Rh}_{α})

$= \{[T_{h'}]: h' \; R \; h \; \& \; h'(\alpha) \neq \mathbf{t}\}$ (by the definition of $\sim$)

$= -\{[T_{h'}]: h' \; R \; h \; \& \; h'(\alpha) = \mathbf{t}\}$ (by the definition of $-$)

$= - S^{Rh}_{\alpha}$ (by the definition of S^{Rh}_{α}).

So the truth-systems $\mathcal{S}^{Rh}$ of logic $\mathcal{L}$ differ from its algebra $\mathcal{A}$: while the algebra $\mathcal{A}$ is a distributive lattice with an orthogonal and a complement operation, on the truth-systems these two monadic operations coincide. Truth-systems are Boolean algebras.

It is interesting that denial *does not* generate an involution on representative algebra $\mathcal{A}$ of logic $\mathcal{L}$ but it *does* generate an involution on the truth-systems. It fails to do so on the logic because when α has no truth-value $\sim\alpha$ is true, so $\sim\sim\alpha$ is false which means these two propositions α and $\sim\sim\alpha$ are not logically equivalent in logic $\mathcal{L}$ which means $[\sim\sim\alpha] \neq [\alpha]$ on $\mathcal{A}$, $\mathbf{a}'' \neq \mathbf{a}$, and so the operation ' representing $\sim$ on $\mathcal{A}$ is not an orthogonal operation. On the truth-systems however things are different. For proposition α is true in h if and only $\sim\alpha$ fails to be true, which means the denial of this denial, $\sim\sim\alpha$, must be true: $h(\alpha) = \mathbf{t}$ iff $h(\sim\alpha) \neq \mathbf{t}$ iff $h(\sim\sim\alpha) = \mathbf{t}$ and so $\alpha \in T_h$ iff $\sim\sim\alpha \in T_h$ which means that $S^{\mathbf{Rh}}_{\sim\sim\boldsymbol{\alpha}} = S^{\mathbf{Rh}}_{\boldsymbol{\alpha}}$ and $- S^{\mathbf{Rh}}_{\boldsymbol{\alpha}} = S^{\mathbf{Rh}}_{\sim\boldsymbol{\alpha}}$ and denial generates a set complement on any truth-system. Each truth-system is a Boolean algebra even though the algebra $\mathcal{A}$ of logic $\mathcal{L}$ is not.

5. Degrees of truth in global $\mathcal{L}$

Degrees of truth are numerical uncertainties, modalities that express "how true" a proposition is in the sense of how many relevant valuations find it true. They are defined using mathematical measures over truth-systems. Because of Result 4.2 above these measures are well-defined.

Defn 5.1: (Rh-degree of truth, $\deg_{Rh}(\alpha)$, μ^{Rh}, D^{Rh}, valuation of degrees $\mathbf{h}^{R}$)

For any α in $\mathcal{L}$, valuation h in H, R a binary relation over H

i) The *Rh-degree of truth*, or Rh-uncertainty, of *α given R and initial h, is* $\deg_{Rh}(\alpha) = \mu^{Rh}(S^{Rh}_{\alpha})$ for μ^{Rh} a measure over the truth-system $\mathcal{S}^{Rh}$.

ii) Numerical connectives D^{Rh} for D in the real interval [0, 1] can be added to the alphabet of logic $\mathcal{L}$ defined by the global definition that for any h' in H, $h'(D^{Rh}\,\alpha) = \mathbf{t}$ iff $D = \deg_{Rh}(\alpha)$, and $h'(D^{Rh}\,\alpha) = \mathbf{f}$ otherwise.

iii) an *uncertain valuation*, or *valuation of degrees* $\mathbf{h}^{R}$ is a many-valued mapping to the propositions defined by setting $\mathbf{h}^{R}: \mathcal{L} \to [0, 1]$ where $\mathbf{h}^{R}(\alpha) =_{df} \deg_{Rh}(\alpha)$

By definition i) degrees of truth are measures over truth-systems of filters related to an initial condition that contain an equivalence class, expressing how many valuation relevant to this initial condition find a proposition true.[x] By ii) these degrees can be considered numerical connectives in global logic that are true when this number is the value of the appropriate measure and false otherwise. By iii) degrees of truth are expressed by many-valued valuations assigning numbers in the interval [0, 1] to propositions, where each proposition is assigned the appropriate degree of truth.

Examples help show the expressive power of these degrees. Consider ordinary propositions such as "Tom is tall", "the road is long", "It is very late". The terms "tall", "long" "late" are all ordinarily imprecise. This means the propositions can be true or false, but their truth-value depends on context, on what other propositions are deemed true. Such imprecise terms can be forced to take truth-values by appeal to some precise definition, but this can lead to paradox. For example suppose that "a tall man" is defined to be "1.80 m or taller". While this introduces precision, allowing truth-values to be assigned to propositions, a paradox arises. Two men of height say 1.7905 and 1.8 are now judged "not tall" and "tall" respectively. Yet this is counter to the meaning of "tall" in ordinary language since such a small difference could never ordinarily distinguish between a man who is "tall" and "not tall". Such a definition is counter to the meaning of vague terms.

In ordinary language propositions based on vague terms often seem not to have a truth-value but instead a logical value that lies on some continuum in between. Degrees of truth achieve just this. "Tom is tall" may take different truth-values even when the same Tom is described, depending on what other propositions are found true. For example if short people are described then "Tom is tall" seems true, while if other much taller people are described this proposition may be false, but in general we may consider it "largely true" if Tom's height is a little more than average. A degree of truth of 0.6 expresses this, being intuitively understood to mean "Tom is tall-ish" or "more tall than short". Formally such numbers can be understood precisely as a measure of a set of related valuations that find "Tom is tall" is true.

The ancient "paradoxes of the heap" are avoided in global logic $\mathscr{L}$ because degrees of truth can be assigned to propositions here as well as truth-values. Gradual change, which causes problems for truth-functional logic similar to those of vague term "tall", can now be described by a gradual change of logical value. In the paradox "Here is a heap of sand" is initially supposed true, then individual grains are removed one by one, until eventually only a few grains are left. At this stage the proposition "Here is a heap of sand" is clearly false, since a few grains is not a heap. The paradox however is in the gradual change. At what stage in this process did the truth-value of this proposition change? Even if some new rule is introduced, for example a definition of a heap as more than 5000 grains, this means a single grain could make the difference between a heap and not a heap, contradicting our notion of this term being imprecise. Even if this is expressed in the truth-functional logic $\mathscr{L}$ with truth-value "gaps", a problem remains.

Global logic however can express multiple valuations and degrees of truth can be assigned to propositions at each stage of the sand removal. This means a gradual real change is described by a gradual change in the degree of truth. Initially when the heap is very large "Here is a heap of sand" is true in every description of this phenomenon, no matter what other truth-values are assigned to other propositions. These might include for example a discussion of aesthetics, of building in the area, of children playing in the sand. In all cases "Here is a heap of sand" is true and so a degree of truth of 1 is assigned. However as the pile reduces in size fewer valuations will find this proposition true. A description of building work using large equipment for example may find this proposition false once it is not extremely large, while a discussion of young children playing in the sand continues to find it true. As the heap gets smaller it follows that fewer valuations find it true. Thus the gradual diminishment of the heap is described by gradual diminishment of the degree of truth, and paradox is avoided.

Different relations among the valuations of logic $\mathscr{L}$ will generate different degrees of truth according to the definition 5.1. Probabilities are one important example of degrees and are discussed in the next few sections below, generated by a formal relation of consistency over H. Other kinds of uncertainty arise from other different relations over H. These may be formally defined but may be contextual, depending on the propositions each valuation finds true. Nuances of ordinary language can be expressed formally in global logic $\mathscr{L}$ that never could be expressed truth-functionally. And unlike

Kripke systems for example, $\mathcal{L}$ can different uncertainties generated from different relations, even in the same complex propositions.

Consider the example p = "Tom is tall" again, and suppose Tom's relatives are very short. This means "Tom is tall" is often true in valuations describing his family. Now consider Tom's attempt to join the national Basketball team. "Tom is tall" will often be false in valuations describing basketball players, because these men are unusually tall. So the contextual degree of truth of p generated by relation F, "family-describing-valuations" is different from the degree of truth generated by B, the "basketball-player-descriptions". This variation by context is expressed in vernacular as "Family-wise Tom is tall but basketball-wise he is short!".

Semantic relations over H allow this to be expressed precisely in global logic. Relation F over H relates all valuations assigning truth-values to descriptions of Tom's male relatives. Relation B relates all valuations assigning truth-values to descriptions of basketball players. The initial valuation in these examples is assumed to be the trivial valuation h_0. Thus two different truth-systems $\mathcal{S}^F$ and $\mathcal{S}^B$, of family- and basketball- relevant valuations respectively, are generated, assigning different degrees of truth to "Tom is tall". Because his family are short, Tom by comparison is tall, and so p is often true in $\mathcal{S}^F$. A high proportion of valuations measured in this truth-system find p true: say $deg_F(p) = \mu^F(S_p^F) = 0.9$, meaning that 9 out of 10 valuations assigning truth-values to propositions about his family find "Tom is tall" is true. On the other hand basketball players are mostly very tall and so fewer valuations in $\mathcal{S}^B$ will find that "Tom is tall" is true. The degree of truth of p in this truth-system is therefore lower, say $deg_B(p) = \mu^B(S_p^B) = 0.3$, meaning that only 3 out of 10 valuations about basketball players find "Tom is tall" is true. The same proposition has different degrees of truth in truth-systems generated by different relations R on H.

In logic $\mathcal{L}$ these different degrees of truth can be combined in a complex proposition. The family-wise degree of truth of p was 0.9 and so the complex proposition $0.9^F p$ is true in h, $h(0.9^F p) = \mathbf{t}$. However the Basketball-wise degree of truth was 0.3 and so $h(0.3^B p) = \mathbf{t}$ as well. These two propositions can be combined in the conjunction $(0.9^F p \wedge 0.3^B p)$ which is also true in h. Alternatively by Defn 9.1 the same degrees can be expressed by uncertain valuations and so for example $\mathbf{h}^F(p) = 0.9$. Similarly since $h(0.3^B p) = \mathbf{t}$ it follows that $\mathbf{h}^B(p) = 0.3$. Both methods express that "Tom is tall" has a family-wise degree of truth of 0.9 while basketball-wise the degree is 0.3. Formally this means that 9 out of 10 "family" descriptions find "Tom is tall is true" while only 3 out of 10 "basketball" descriptions do so. Later, in section 9 these degrees are also expressed in terms of "fuzzy sets".

Probabilities are developed as degrees of truth in the next sections. It is useful to use the natural correspondence of Defn 3.5 above express degrees of truth in terms of propositions or of valuations. Already it has been noted that any relation among the valuations of $\mathcal{L}$ generates a corresponding relation among their characteristic propositions, and this also generates a relation among the corresponding equivalence classes. In fact degrees of truth can be expressed with propositions as initial condition, and to be assigned to valuations as well as propositions using these correspondences.

Defn 5.2: (relation R over H, $\mathcal{L}$ or $\mathcal{A}$; $deg_{R\alpha}(\beta)$, $deg_{Rh}(h')$)

For any relation R over H and propositions α, β of $\mathcal{L}$

i) h_α R h_β iff α R β iff $[T_{h_\alpha}]$ R $[T_{h_\beta}]$.

ii) $deg_{R\alpha}(\beta) =_{df} \mu^{R\alpha}(S_\beta^{R\alpha}) = \mu^{Rh_\alpha}(S_\beta^{Rh_\alpha})$.

ii) $deg_{Rh}(h') =_{df} deg_{Rh}(\beta_{h'})$.

By i) any bivalent relation R among valuations generates a corresponding relation among their characteristic propositions or equivalence classes. By ii) a degree of truth can have a proposition as an initial condition, it is the degree with the initial characteristic valuation. By iii) a degree of truth can be assigned to a valuation, this is the degree of truth of its characteristic proposition. These correspondences will be particularly useful in later discussion of probabilities.

6. Probabilities

Probabilities are degrees of truth in a logic that describes reality. The nature of reality is not considered here, nor is the nature of description or questions of how it is that language can describe reality. These topics lie outside logic in metaphysics. Logic instead takes the truth-values true and false as fundamental and the "meaning" of a proposition is expressed when a truth-value is assigned. If propositions of the language are descriptions, then this meaning is assumed to be a description of reality, so valuations express descriptions.

Probabilities express how "likely" a proposition is to be a true description of reality given some initial description of the system. This means probabilities are degrees of truth generated by a "successor" relation among valuations, which holds between one description of reality and another that might next be used to describe the same real system. The probability of a proposition is thus a measure of how many valuations that might describe the same reality as the initial condition, find a proposition true.

To formally define probabilities the "successor" relation over H must be identified. Recall in section 3 above *truth-sets* were introduced as the set of propositions a valuation finds true (defn 3.3). In section 4 a partial ordering, inclusion or agreement among valuations was generated by set inclusion among truth-sets: one valuation h *includes* another h', when it *agrees* with its truth-value assignments, which means that every proposition true in h' is true in h, a definition we now call strong consistency (defn 4.1). In this same set of definitions a valuation was called *maximal* when it is maximal with respect to this relation, ie. when its truth-set is maximal with respect to set inclusion which means that no other valuation makes the same truth-value assignments and yet makes more. Some new terms will also be useful.

Defn 6.1: ((non-)contradictory, strong- weak- consistency; bivalent, classical logic; compatible, incompatible propsitions)

For valuations h, h' in H and propositions α, β of $\mathcal{L}$

i). Valuations h, h' are *contradictory* if for some α, $h(\alpha)=\mathbf{t}$ but $h'(\alpha)=\mathbf{f}$ and are *non-contradictory* (*weakly consistent)* h W h' if there is no such α in $\mathcal{L}$.

ii) Valuation h' *agrees with* (is *strongly consistent* with) valuation h, $h \subseteq h'$, if every proposition true in h is true in h', ie. if $T_h \subseteq T_{h'}$.

iii) Valuation h is *bivalent* if it assigns a truth-value to every proposition of the logic, ie. $C_h = L$.

iv) Logic $\mathcal{L}$ is *bivalent* if every valuation h in H is bivalent and is *classical* if all its maximal valuations are *bivalent*

v) Propositions α, β are *compatible* if there for some h $\alpha, \beta \in C_h$ and are *incompatible* otherwise.

By i) *contradictory* valuations make conflicting truth-value assignments to the same proposition, while *non-contradictory* valuations do not. By ii) a valuation is *strongly consistent* with another if it agrees with it, making at least the same truth-assignments, as defined earlier. The new name is introduced because this property is stronger than the first: clearly any valuation that agrees with another does not contradict it but the converse is not the case, and so non-contradiction is called *weak* and agreement *strong consistency*. By iii) a *bivalent valuation* assigns a truth-value to every proposition in the logic. By iv) a *bivalent logic* has all valuations with this property, while a *classical logic* has maximal valuations bivalent. This means a logic may be classical but not bivalent, non-standard terminology that will be justified later when classical logic is shown to have classical probabilities. Lastly two propositions are *incompatible* when no valuation assigns truth-values to them both, and are *compatible* otherwise.

Clearly all propositions of a classical logic will be mutually compatible, since a bivalent valuation assigns truth-values to all propositions. In general however a logic lacks bivalent valuations and therefore has some propositions that are mutually incompatible, that cannot be assigned truth-values in the same valuation. This turns out to be a crucial property that determines the *system* of probabilities used by a logic.

If probabilities were degrees of truth based on *strong* consistency, then only valuations that *agree* with initial truth-assignments could be included in the measured sets. But this means a valuation assigning truth to a proposition incompatible with initial truths would be ignored. This seems inappropriate, since where a logic includes incompatible descriptions of the same reality they should surely be considered in its probabilities. So strong consistency is rejected as the successor relation for probabilities, and this is assumed to be weak consistency instead. It is also assumed that a probability measures only *maximal* valuations, for only the "fullest" descriptions of a real system need be measured in a probability: since every valuation is by definition included in a maximal one only "partial" descriptions will be ignored.

A probability then, is a measure of the set of maximal descriptions that do not contradict an initial valuation, and that find a proposition true, an uncertainty derived from relation W over H:

Defn 6.2: (h-probability space $\mathcal{P}^h$, $prob_h(\alpha)$, simple probability space $\mathcal{P}$)

For any h in H, α in $\mathcal{L}$, and relation W of weak consistency:

i) The *h-probability-space* $\mathcal{P}^h$ of logic $\mathcal{L}$ is the truth-system generated by weak consistency W and initial condition h, ie. $\mathcal{P}^h = \mathcal{S}^{Wh}$.

ii) The *probability of proposition* α given initial h is a measure over this truth-system, ie. $prob_h(\alpha) =_{df} deg_{Wh}(\alpha) = \mu^{Wh}(S_\alpha^{Wh})$, a measure over the *h-probability-space* of the set of maximal valuations in H that do not contradict h and that find α true.

iii) The *simple probability space* $\mathcal{P}$ of any logic $\mathcal{L}$ is the simple truth-system for probabilities, ie. $\mathcal{P} =_{df} \mathcal{P}^h$ for h = h_0 the trivial valuation.

So by this fundamental definition a *probability space* $\mathcal{P}^h$ is a truth-system derived from maximal valuations that do not contradict initial h: a system of sets of maximal valuations in H that do not conflict with this initial description and that find each proposition true. Since a probability space is always derived from the same relation W of non-contradiction among maximal valuations, this can be omitted from the formalism. A *simple* probability space $\mathcal{P}$ has a trivial initial condition and so is a system derived from the set of filters representing all the maximal valuations of H.

In a classical logic weak consistency among maximal valuations coincides with strong consistency. This means the simple probability space has special significance in a logic that is classical.

Result 6.1: Every probability of a classical logic $\mathcal{L}$ can be expressed in terms of the simple probability space.

<u>Proof</u>: In a classical logic $\mathcal{L}$ all maximal valuations are bivalent, (by defn). But bivalent h' of $\mathcal{L}$ is non-contradictory with h if and only if it agrees with h: <= is obvious; to show => suppose the contrary, ie. that h' is bivalent, h' W h but h $\not\subset$ h' and so some α is true in h but not in h'; but because h' is bivalent this means α is false in h' which contradicts the assumption that h' W h, and so h $\subset$ h'. It follows that

$prob_\alpha(\beta) = deg_{\subseteq\alpha}(\beta)$ (because in this special case W = $\subseteq$ by above)

$= \mu^{\subseteq\alpha}(S_\beta^{\subseteq\alpha})$ (by the definition of deg)

$= \mu^{\subseteq\alpha}(S_{\alpha\wedge\beta})$ (by properties of $\subseteq$)

$= \mu(S_{\alpha\wedge\beta}) / \mu(S_\alpha)$ (by properties of $\mu^{\subseteq\alpha}$ where μ is a measure over the simple truth-system $\mathcal{S} = \mathcal{P}$).

So where all maximal valuations are bivalent the simple probability space of a logic can express all its probabilities.

Because they are bivalent, the maximal valuations of a classical logic $\mathcal{L}$ find either α or its negation $\neg\alpha$ true for every proposition α in $\mathcal{L}$, which means that the truth-sets of these valuations are represented by filters of algebra $\mathcal{A}$

of $\mathscr{L}$ with the ultrafilter property, (Defn 4.3). That is, the truth-sets are represented by Boolean ultrafilters of a Boolean algebra $\mathscr{A}^*$ that represents the bivalent version of logic $\mathscr{L}$ and so in this case $\mathscr{P}$ of logic $\mathscr{L}$ coincides with the Stone space of Boolean algebra $\mathscr{A}^*$ representing the bivalent version of logic $\mathscr{L}$. It appears that the probabilities in this case have the traditional logical foundation as measures over a field of ultrafilters representing the bivalent valuations in which a proposition is true. However there is one key difference between the traditional and present view, which is that the algebra $\mathscr{A}^*$ is *not* assumed to represent the logic $\mathscr{L}$ because this logic is not bivalent. Instead $\mathscr{L}$ is represented by the algebra $\mathscr{A}$ discussed in section 2, a distributive lattice with orthogonal and complement rather than a Boolean algebra where orthogonal and complement coincide. So although every logic has a simple probability space, only in bivalent logic will this be the Stone space of its algebra.

By Result 6.1, probabilities have the same definition in any logic $\mathscr{L}$ but they generate very different systems according to whether this logic is classical or not. Where the logic is classical all maximal valuations are bivalent, all propositions in logic $\mathscr{L}$ are mutually compatible, and all the logic's probabilities can be expressed on the simple probability space. However in general maximal valuations are not bivalent and so there are incompatible propositions and so the simple probability space lacks this classical role. In general probabilities are conditional in the strong sense that the very probability space over which they are defined, depends on the initial condition. This means a family of different probability spaces is required to express all of a logic's probabilities. This key difference between classical and non-classical probabilities accounts for the Hilbert Space representation of quantum mechanics, as will now be discussed.

7. Probabilities in mechanics

Theories of both classical and quantum mechanics use *magnitudes* including *position* and *momentum* to describe real systems. Each theory T of mechanics has a set M_T of magnitudes to describe a reality, where each magnitude m in V_m has a *value-set* V_m of real numbers, the allowable values for this magnitude on the reality described, according to this theory.

Simple propositions in mechanical theories therefore have a common form. They can be expressed as ordered pairs (m, Δ), where m is a magnitude in M_T and Δ a Borel subset of the value-set V_m. Assuming each subset Δ is Borel ensures that set operations among them will be well-defined as well as *atomic* propositions of form (m, r) for $r \in V_m$ the most "precise" m-propositions of the theory. Atomic proposition (m, r) asserts "The value of magnitude m on this reality is r".

Set relations among subsets of V_m generate relations within each m-system of propositions that are assumed expressed by logical connectives. It is assumed for example if p = (m, Δ) and p' = (m, Δ') that the set union $\Delta \cup \Delta$ is expressed by logical disjunction, ie. $p \vee p' = (m, \Delta \cup \Delta')$. This follows from the assumption that each m on the system is characterised by an infinitely precise value. For if (m, r) is true for some r in V_m, then r is in $(\Delta \cup \Delta')$ iff it is in either Δ or Δ' and so $(m, \Delta \cup \Delta')$ is true iff either one of p or p' must be true, which is the defining property of disjunction. Similarly conjunction is assumed to express set intersection, $p \wedge q = (m, \Delta \cap \Delta')$ because the value of m is in this intersection only if it is also in each of the two subsets and so $(m, \Delta \cap \Delta')$ is assumed true iff either p or p' is true, a condition that characterises conjunction. Implication similarly expresses set inclusion since when $\Delta \subset \Delta'$ then if r is in Δ it is also in Δ' and so $p \supset p'$ in this case is true. Finally negation expresses the set complement since the value of m is in set Δ iff it is outside the set-complement $(V_m - \Delta)$ and so $p = (m, \Delta)$ is true iff $\neg p = (m, V_m - \Delta)$ is false.

Both classical and quantum theories therefore have simple propositions of similar form, and these form traditional subsystems of m-propositions for each magnitude m in the theory, with traditional logical connectives generated by set operations on each value-set. One consequence is that when a valuation of the logic of either kind of theory finds any atomic m-proposition true it also assigns truth-values to all the other m-propositions. For when some (m, r) is true in h, then (m, r') must be false in h for every other atomic proposition, $r \neq r'$ by the assumption that negation expresses the set complement on V_m, and every other simple proposition (m , Δ) will be true iff r is in Δ and will be false otherwise. Both kinds of theory assume an atomic truth-value means truth-values are assigned to all the m-propositions.

The logic of both classical and quantum theories is the logic $\mathcal{L}$ discussed above, founded on primitive truth-values and the traditional Laws of Thought. So it is neither the form of simple propositions nor the logic used to combine these, that distinguishes classical from quantum theories. What is different in either kind of theory are the mechanical laws relating magnitudes in M_T and hence the propositions the magnitudes generate. In classical mechanics all magnitudes are compatible in the sense that their simple propositions are all mutually compatible, they can all be assigned truth-values in a common context. The logic of any classical theory of mechanics has maximal valuations that are bivalent, assigning true or false to all atomic proposition of the theory, and hence to all other propositions as well. However quantum theories have magnitudes that are incompatible in the sense that they generate simple propositions which are incompatible. According to Heisenberg's Uncertainty Principle no two atomic propositions in quantum theory of position- and momentum- magnitudes can be assigned truth-values in the same valuation. The logic of quantum theories have incompatible propositions and hence maximal valuations that are not bivalent. The logic of these theories is therefore not classical in the sense of Defn 6.1.

This difference between classical and quantum theories of mechanics means that by Result 6.1 there is a fundamental difference in their systems of probabilities. According to this result probabilities of classical mechanics can all be expressed over the simple probability space of the logic of this theory, which means the familiar representation of probabilities holds. However since quantum logic has incompatible propositions and maximal valuations that are not bivalent, this special result does not hold for these theories. Quantum probabilities *cannot* be expressed in just the simple probability space because the successor relation in this case is weak not strong consistency. These probabilities are strongly conditional in the sense that the probability space over which they are defined, depends on the initial condition. Quantum probabilities require a very different and non-classical mathematical expression.

All probabilities in mechanics are conditional on the *physical measurement* of a magnitude. *Physical measurement* is not to be confused with a *mathematical measurement*: the first is a procedure, in principle describable, that results in truth being assigned to some non-trivial simple proposition of the magnitude concerned; while the second is a mathematical function over a Boolean field of sets. Physical properties of measurement are not considered here. The logical property of physical measurement of magnitude m of the theory, is that immediately afterwards some non-trivial m-proposition is found true, some (m, Δ) where Δ is neither the empty set nor the entire set V_m. The "most precise" such m-proposition (m, Δ) resulting from physical measurement is that with the "smallest" set Δ with respect to set inclusion, and this is the *measurement outcome*. Where a measurement outcome is an atomic proposition (m, r), then the physical measurement is called *ideally accurate* and in this case value r is "the value of m" on this system according to the theory.

Classical theories of mechanics, having bivalent maximal valuations, use the simple probability space $\mathcal{P}$ to express every probability. This measure space is a field of Boolean ultrafilters containing each equivalence class, representing the maximal bivalent valuations of the theory in which the corresponding propositions are true. This also naturally corresponds to the space of n-tuples of real numbers $(r_1, r_2, \ldots r_n)$, the "points" in n-space corresponding to precise values of all the primitive magnitudes of the theory, ie. these are the values in the atomic m-propositions true in each bivalent valuation of logic $\mathcal{L}_T$ for each primitive magnitude m in M_T. This space of points is the traditional "phase space" of classical mechanics, and is often considered an underlying space of "properties" or "events" over which the theory's probabilities are defined. This can also be regarded as a Euclidean vector space, where vectors over the space represent classical probability assignments.

Quantum theories of mechanics however have no such representation, because they have incompatible magnitudes generating incompatible propositions and so lack bivalent maximal valuations, ie. they are not classical. In these theories the simple probability space of the logic can be defined but cannot express all the theory's probabilities and instead different probability spaces are required for different initial conditions. There is therefore no corresponding space of Boolean ultrafilters or "phase space" of points representing precise values for all the magnitudes in bivalent valuations. It has seemed that in lacking this classical phase space quantum theories lack underlying "events" or "properties". However we must remember that logic is not concerned with "events" or "properties" but with true descriptions. The probabilities of quantum theories, like those of classical mechanics, are measures of sets of maximal filters representing maximal truth-sets, expressing "how many" consistent valuations find a proposition true. In quantum theories these cannot be expressed on a single probability space and so it follows that the system of probabilities requires a mathematical representation that is bound to be radically non-classical.

In fact quantum probabilities are expressed using Hilbert Space, a (usually) infinite-dimensioned, complex-valued, inner product vector space. *Observable operators **A, B**...* over a quantum theory's Hilbert Space are linear, Hermitian operators used to express the magnitudes of the theory. Quantum "states" are normed unit vectors of the Hilbert Space that express quantum descriptions of reality by generating probabilities using the inner product operation. In the simplest case operators acting on these vectors satisfy the eigenvalue equation $\boldsymbol{A}\alpha_i = a_i\alpha_I$ so that applying the operator to an eigenvector simply alters it by a scalar. In fact this is a simplification since mostly a more complex mathematical expression is required in terms of vector spaces not eigenvectors, however in principle the issues are similar so this case is used to simplify discussion. The set of eigenvectors $\{\alpha_i\}$ that satisfy this equation make up an *eigenbasis* for $\boldsymbol{A}$, which means this is a complete, orthonormal set that represents this operator. Because the operator is Hermitian, the corresponding set of *eigenvalues* $\{a_i\}$ are real.

By Born's Rule a physical measurement of magnitude A on a system initially described by quantum state ψ, leaves the real system in an eigenvector of observable operator $\boldsymbol{A}$ representing this magnitude, and the corresponding eigenvalue is the "value of A" on the system according to this quantum theory. Furthermore the probability that particular eigenvector α_k of $\boldsymbol{A}$ describes the system after this physical measurement, ie. the probability that a_k is the value of A, is derived from the inner product (α_k , ψ) of these two vectors. This probability is $|(\alpha_k , \psi)|^2$, a value sometimes called the "component" of ψ in the direction of α_k, or the "projection" of vector ψ onto α_k. Born's Rule not only asserts this scalar is the probability but also that this is the most information that can be derived from initial quantum state ψ about the probability of eigenvector α_k describing the reality after the physical measurement of A, ie. the probability that a_k is the value of magnitude A after this physical measurement.

The new state α_k predicts this same value a_k with certainty if the physical measurement is immediately repeated. For quantum states have unit norm, which means $|(\alpha_k , \alpha_k)|^2 = 1$, and so the probability of finding α_k again after another measurement of A is a certainty. However any other eigenvalue a_j of A has probability 0 in this case because the eigenvectors are also mutually orthogonal, which means $|(\alpha_k , \alpha_j)|^2 = 0$ for $k \neq j$. So quantum state α_k predicts value a_k of A with certainty given an immediate repeat of the measurement of A, and predicts any other value a_j of A with zero probability.

Eigenvectors of observable operators are the so-called "pure states" of quantum mechanics, while "mixed states" are other normed vectors that also generate probabilities but are not eigenvectors of an observable operator, and so do not predict any eigenvalue of an operator with certainty.

8. Quantum non-classical peculiarities

Quantum probabilities in Hilbert Space will now be expressed in logical terms so their peculiarities can be compared with non-classical logical probabilities. In fact these peculiarities are the same.

The "pure states" of quantum theory are eigenvectors of observable operators, whose eigenvalues are the allowable values of the magnitudes they represent on the system described. Each eigenvalue a_k therefore is a value in the value-set V_A of magnitude A in this theory T, and so corresponds to an atomic A-proposition $p_k = (A, a_k)$ in the logic $\mathcal{L}_T$ of T. The pure state α_k expresses a description in which eigenvalue a_k is predicted with certainty, so α_k corresponds to the characteristic valuation of p_k, which finds p_k and all its logical consequences true. The "mixed states" of a quantum theory also generate probabilities but these states are not eigenvectors of an observable operator and so do not predict an eigenvalue with certainty. Mixed states correspond to valuations of logic $\mathcal{L}_T$ that are not characteristic for an atomic proposition.

Where ψ is some initial quantum state, $\psi = h$, then by the discussion above the quantum probability that α_k describes the system after a physical measurement of A given initial ψ, ie. that value a_k is found true of A after this measurement, is $|(\alpha_k , \psi)|^2$. This therefore is the logical probability $prob_h(p) = \mu_h(h_p)$, which is a measure of the set of maximal valuations not contradicting $h = \psi$, that find $p = (A, a_k)$ true. The quantum representation in Hilbert Space seems a natural way to represent these non-classical probabilities. For projections $|(\alpha_k , \psi)|^2$ along the eigenvectors (or strictly speaking over subspaces of the Hilbert Space associated with the observable operators) seems an appropriate expression for the strongly conditional non-classical measures over different probability spaces. This is however not only a natural way to express these probabilities, but one that shares the key peculiarities of the logical probabilities. The peculiarities familiar from quantum probabilities are also peculiarities of the logical probabilities when a logic is not classical, ie. when there are incompatible propositions so that maximal valuations are not bivalent.

One quantum peculiarity is that probabilities do not commute because the inner product operation defining them does not commute. Given initial state ψ, the probability that another state ϕ describes the system after a physical measurement is not the same as the probability of ϕ given initial ψ, because $(\psi, \phi) \neq (\phi, \psi)$ in general. This has seemed peculiar because classical probabilities are expressed in terms of a corresponding joint probability on phase space, which does commute. Yet the lack of commutation is shared by the logical probabilities in general. Where a logic is classcial all probabilities are expressed on the simple probability space by Result 6.1, and the probability $prob_h(h')$ is expressed by a corresponding joint probability $prob(\alpha \wedge \beta)$ in a subspace of the simple measure space, where α, β are wffs characteristic for valuations h, h' respectively. However for corresponding probabilities in a logic that is not classical commutation fails. For here $prob_h(h')$ is a measure over the valuations that do not find α false and find β true, while $prob_{h'}(h)$ measures valuations that do not find β false but find α true. Where these are incompatible propositions the measures are over different measure spaces of different sets and do not generally coincide so $prob_h(h') \neq prob_{h'}(h)$ and probabilities do not commute.

The irreducibly statistical nature of quantum probabilities is also a feature of the logical probabilities. As discussed earlier, quantum probability $|(\alpha_k , \psi)|^2$ is 1 only when α_k coincides with initial ψ, in which case all other eigenvectors of $\boldsymbol{A}$ have probability 0. But even in this case irreducibly statistical predictions are made about the value of an incompatible magnitude B. If β_j is an eigenvector of observable operator $\boldsymbol{B}$ representing this magnitude, then the inner product $(\beta_j , \alpha_k) \neq 1$ and $\neq 0$ so probabilities conditional on ψ are irreducibly statistical. Logical probabilities share this property. Where atomic proposition $p = (A, a_k)$ is an outcome of a physical measurement of A, then a_k is *certain* to be true if this physical measurement is repeated. For a successor valuation of h_p cannot find p false which means that no other atomic A-proposition can be true and so $prob_p(p) = 1$ and $prob_p(p') = 0$ for every other atomic proposition p', exactly as in quantum theory. However only irreducibly statistical predictions can be made about values of incompatible magnitudes because if $q = (B, b_k)$ is incompatible with p, then by definition it does not have a truth-value in h_p and so h_q does not contradict h_p and a successor valuation of initial valuation h_p can ind q true, so $prob_p(q) \neq 0$. Because some other valuation that agrees with h_p can be its successor and will not find q true since p and q are incompatible, it follows that not *every* successor of p finds q true and so $prob_p(q) \neq 1$. These logical probabilities, just like quantum probabilities, are in general irreducibly statistical.

Measurement-dependence is also a peculiarity of both quantum and logical probabilities. By Born's Rule, initial state ψ changes to an eigenvector α_k of $\boldsymbol{A}$ after an ideally accurate physical measurement of A. However if magnitude B were subject to physical measurement instead of A, then the change would be to an eigenvector of observable operator $\boldsymbol{B}$ representing B, say β_j instead. These two different vectors, α_k or β_j, generate different predictions because $|(\varphi, \beta_j)|^2 \neq |(\varphi, \alpha_k)|^2$ in general. So the choice of magnitude to subject to a physical measurement generates different quantum probabilities. The logical probabilities similarly depend on physical measurement even though only *logical* properties of this process have been considered here. Given some initial state h the choice of magnitude to measure determines the probability space generating subsequent predictions. After a physical measurement of A for example, probabilities are not conditional on h but on some A-outcome q, while after a physical measurement of B h changes to a valuation that finds a B-outcome true. Where these different measurement outcomes are mutually incompatible resulting probabilities will be different for $prob_q(\gamma) \neq prob_p(\gamma)$ where p and q are incompatible. On the left is a mathematical measure of the set of maximal valuations that do not contradict A-outcome q and find γ true, while on the right is a measure of the maximal valuations that do not contradict B-outcome p and find γ true. These are measures of different sets on different measure spaces and so will be distinct.

Finally sequences of physical measurements in quantum theory appear peculiar because "Bell-type inequalities" fail even though they are apparently derived from self-evident equalities. Consider a sequence of measurements of magnitudes A, B, C, each with just two values represented by + or – . For simplicity lets call these magnitudes *size*, *colour* and *weight*, with values big or small, black or white, heavy or light respectively. Classically a value of any of these magnitudes can be expressed in terms of others, for example $(A+) = (A+ \wedge B+) \vee (A+ \wedge B-)$ an equality supposed self-evident because only two possible values are allowed for B. This might read "All big bodies are either white or black" for example. From this equality various inequalities can be derived, including $prob((A+) \wedge (B-)) \leq prob(A+)$, "The probability of finding A big and white is less than or equal to the probability of finding A is big", which appears to follow from the earlier equality. Similar expressions and inequalities can be derived about the other values of other magnitudes and by combining such expressions about more than three different magnitudes the complex "Bell-type inequalities" are derived.

However though these seem derived from self-evident equalities the Bell type inequalities fail in quantum theories, a failure supported by experiment. For example this is shown in experiments concerning spin in three mutually incompatible directions a, b and c, expressed in quantum theory by incompatible magnitudes A, B, C each with just two possible values "up" or "down". Observable operators ***A, B, C*** representing these magnitudes on Hilbert Space have just two eigenvectors each, $\alpha+$ of ***A*** for spin "up" in direction a for example, β^- for "down" in direction B. Bell type inequalities in such a case simply do not hold in quantum theory, a failure that has been confirmed by experiment, leading to claims of non-local effects and "entangled" quantum properties.

Yet similar inequalities also fail for sequences of logical probabilities in general, suggesting logical rather than real entanglements. These inequalities fail because the supposedly "self-evident" expressions from which they are derived, also fail. In a logic with incompatible propositions the supposedly self-evident equalities simply do not hold. It is not the case in such a logic that $(A,+) = ((A,+) \wedge (B,+)) \vee ((A,+) \wedge (B,-))$. If "size" and "colour" for example have incompatible propositions then "All big bodies are either white or black" no longer holds. For since the simple propositions are incompatible they have no common context, so when (A, +) is true in h, then (B,+), (B, –) are both unassigned. This means because since colour is incompatible with size in our example, when "A is big" is true then neither "A is black" nor "A is white" is true, even though these are the only two colour options. Another way of expressing this is to say that Excluded Middle does not hold for negation in such a logic even in the maximal valuations, because these are not bivalent. So ((B, +) ∨ (B, −)) is not a logical truth and so ((A, +) ≠ ((A, +) ∧ ((B, +) ∨ (B, −)) and this means the distributive law cannot be used to derive ((A, +) ≠ ((A, +) ∧ (B, +)) + ((A, +) ∧ (B, −)). Inequalities generated by these expressions therefore also fail, so prob((A, +) ∧ (B, −) ≰ prob(A, +). More complex "Bell-type Inequalities" involving three magnitudes in a sequence will also fail.

This failure is not evidence of "entangled properties", nor does it mean there are non-local effects, nor is it evidence that the logical law of distribution fails in "quantum logic".[xi] Indeed there is no paradox at all in this failure so long as terminology is appropriately logical. The failure of the inequalities shows at most an "entanglement" of descriptions when propositions are incompatible and so there are no bivalent valuations. The one-one correspondences classically assumed between maximal valuations and realities is gone. More than one incompatible maximal description now describes the same reality, so changes can occur in a sequence of descriptions that may not describe real change. There is also information loss as a result of the weak consistency relation, which means that correlations for example that might hold initially on a system may be lost in a sequence of descriptions, even though the system may not have changed. Such a loss of information is also evident in quantum theory.

Disquiet expressed by Einstein about quantum theory is to a limited extent supported by this logical analysis. The use of incompatible descriptions, and corresponding lack of bivalent valuations, weakens descriptive power in the sense that propositions cannot be assumed to express "properties" because these cannot be combined in maximal descriptions. However the claim in the Einstein, Podolsky, Rosen paper that quantum theories are "incomplete" is not justified. At the very outset of their paper these authors make the following assertion:

> *"In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty without disturbing the system…"*

This assertion about "elements of reality" has no place in physics or in logic. It is a claim of metaphysics resembling medieval "proofs" for the existence of God. The strongest claim that can be justified is that a theory without bivalent valuations is inadequate in the sense that descriptions cannot be combined into complexes we commonly call "properties". In such a theory descriptions cannot always be combined, and a change in description may not indicate a change in reality. Both can be considered weaknesses in description, though not an "incompleteness" in the strong metaphysical sense of EPR. After all, this may be the only theory possible of very small reality. However equally this may not, and if there were a theory of the very small will all compatible descriptions then descriptions could always be combined, and probabilities would be classical, expressed on a single probability space without irreducible statistics and measurement dependence. Such a theory could use the term descriptions for "properties" or "events" without paradox. Surely if well supported by experiment, such a theory would be preferable to quantum mechanics.

Work by Kochen and Specker and others has shown that quantum theory is not "incomplete" in the sense that it could be "completed" by the introduction of new magnitudes or "hidden variables" that would allow bivalent valuations. It is now clear that adding new descriptions, via new magnitudes and values cannot turn quantum into a classical logic. A new theory of the very small will therefore use different simple propositions to describe the same realities, related in different way. These might not be "mechanical" propositions in the strong sense that is assumed in classical mechanics, ie. each magnitude m of the theory may not have infinitely "accurate" values for position and momentum. This assumption imposes a structure on descriptions that perhaps prevents the logic being classical.

The logical analysis of probabilities has shown at least that quantum theory may use meaningful descriptions and probabilities defined exactly as in classical mechanics, degrees of truth generated by weak consistency. in each case these are well-defined mathematical measures over Boolean fields of sets of propositions that are true. A probability in quantum theory as in any other logic, expresses its "likelihood" in the sense of how many valuations consistent with an initial condition find it true. However though these definitions are the same in either theory, the systems of probabilities must be represented mathematically in very different ways. Where a logic lacks bivalent valuations one single probability space cannot express all the strongly conditional probabilities. The Hilbert Space representation of quantum probabilities now seems natural, and classical probabilities an important special case.

9. Fuzzy sets

Finally predicates are added to the language of logic $\mathcal{L}$, allowing propositions to express set theory. While the truth-functional logic $\mathcal{L}$ expresses "crisp" sets of normal set theory, degrees of truth of global logic $\mathcal{L}$ provide a new foundation for "fuzzy" sets.

Formally the alphabet of logic $\mathcal{L}$ is extended to include *predicate* variables P, Q…, as well as *individual variables* x, y… and *individual constant* symbols a, b... Rules of Formation include those of the propositional logic, (see Defn 1.3 above), as well as a rule for predicates.

Defn 9.1: (Rule of Formation for predicates)

> If P is a predicate in L, x an individual variable and a an individual constant then Px and Pa are wffs in the predicate logic of $\mathcal{L}$

The predicate logic $\mathcal{L}$ has Rules of Formation for the truth-functional connectives $\wedge$, $\vee$, $\supset$, $\equiv$, $\neg$, $\sim$ see tables 1.1 and 1.2 above, and in addition propositions Px, Qa, are well-formed formulae, understood as "x is P" or "a is Q" respectively.

A "crisp" set is identified with its members, and has a two-valued membership function. There is a natural relation between truth-values assigned to predicate propositions, and the binary membership function so each valuation of logic $\mathcal{L}$ defines a "crisp" set associated with any predicate.

Defn 9.2: (Crisp set $\mathbf{P_h}$)

For P a predicate in the language of logic $\mathcal{L}$, h in H of $\mathcal{L}$,
The *crisp set* P_h generated from P by h has bivalent membership function $X_{Ph} \rightarrow \{1, 0\}$, where $X_{Ph}(x) = 1$ iff $h(Px) = \mathbf{t}$ and in this case $x \in \mathbf{P}_h$ and where $X_{Ph}(x) = 0$ iff $h(Px) = \mathbf{f}$ and in this case $x \notin \mathbf{P_h}$.

So truth-value assignments by h to Px thus generate a bivalent membership function defining crisp set $\mathbf{P}_h$. The truth of a predicate proposition Px corresponds to value 1 of X_{Ph} and the membership of x in the set while the falsity of Px corresponds to value 0 and exclusion of x from the set.

Zadeh, in the 1960's, introduced new "fuzzy sets" with many-valued membership functions. By now such sets are widely used and have had much mathematical attention. However their logical foundations remain obscure since like many-valued logic discussed earlier, the membership-values in the real interval [0, 1] are unexplained. Extreme values 1 and 0 may coincide with ordinary set membership or set exclusion respectively, but the nature of the intermediate "fuzzy" values is unexplained. These values indicate by their proximity to the extremes how "close" they are exclusion or set membership but they are simply assumed as primitive so their logical foundation remains obscure.

The fact that these "fuzzy" values are intuitive adds to the mystery. A predicate like "tall" discussed in section 5, is ordinarily imprecise. The claim that "Tom is tall" is 0.6 true, indicates as earlier discussed that this proposition is nearer true than false and so Tom is "tall-ish", "more tall than short". In a similar way we understand "Tom is a 0.6 member of the set of tall men" to mean that Tom is nearer membership than exclusion to this set. However the "fuzzy" set memberships like 0.6 are not derived but simply assumed as primitive, leaving a foundational problem for these sets. Many-valued propositional logics including the Lukasiewicz systems have been proposed as "fuzzy" logic expressing "fuzzy" sets, but this does not make the foundational problem clear. For their logical values are not explained in terms of primitives. Furthermore truth-functional connectives seem unable to express uncertainties, failures which lead Kripke to devise alternative "modalities". In the logic of fuzzy sets implication is particularly problematic, with truth-functional conditions apparently inadequate to express the "fuzzy implications" that arise when both premise and conclusion may be uncertain.[xii]

Global logic presents a logical foundation without these problems. Connectives are defined by multiple valuations from set H, allowing rich contextual modalities to be expressed as discussed above. *Degrees of truth* provide values for many-valued *uncertain valuations* (Defn 5.2), producing a many-valued logic that is not truth-functional and is capable of expressing probability theory as the previous sections show. The values in this logic are not primitive but are instead derived from traditional truth-values true and false. The logic also has a natural sense of "fuzzy" implication.

Defn 9.3: (uncertain inference $=>_R$, uncertain set $\mathbf{P_h}$)

For P a predicate in language L of logic $\mathcal{L}$, α, β wffs of $\mathcal{L}$, valuation h in H and relation R over H.

i) The *uncertain (fuzzy) set* $\mathbf{P_{Rh}}$ generated by P and h and R has membership function $\mathbf{X}_{PRh}$ (x) –>[1, 0], where $\mathbf{X}_{PRh}$ (x) = $\mathbf{h}^R(Px) = deg_{Rh}(Px)$For any h in H of $\mathcal{L}$ and relation R over H and $\mathbf{h}^R(\alpha) =_{df} deg_{Rh}(\alpha)$

ii) A new connective, *uncertain inference according to R,* $=>_R$, is defined by the many-valued valuation rule that $\mathbf{h}^R (\alpha => \beta) =_{df} deg_{R\alpha}(\beta)$

So by i) the many-valued memberships of an uncertain set are measures over relevant truth-systems of the predicate propositions. The membership functions of a fuzzy-set generated by predicate P, valuation h and relation R, are degrees of truth of the corresponding predicate proposition in valuations related by R to h. If we recall that $\mathbf{h}^R(\alpha) =_{df} deg_{Rh}(\alpha)$ by Defn 5.2, then this is the value of an uncertain valuation $\mathbf{h}^R$. By ii) the uncertain inference of β from α according to relation R, is the degree of β according to R, given initial condition α, a measure of the valuations related to α in which β is true. Clearly many other different conditionals can be defined in the global logic $\mathcal{L}$

An example helps to understand this generalisation from crisp to uncertain set. Let a represent the individual Tom, and Q the predicate "tall", so Qa is the proposition "Tom is tall". According to Defn 9.2 above if Qa is true in h, then $a \in Q_h$, "Tom is in the set of tall men according to h". If Qa is false in h, then $a \notin Q_h$, "Tom is excluded from the set of tall men according to h". The global logic $\mathcal{L}$ retains this relationship with crisp sets but in addition *uncertain* sets can be derived from degrees of truth of these predicate propositions. The uncertain set $\mathbf{Q_{Rh}}$ has *many-valued* membership function $\mathbf{X}_{RhQ}$ expressing a contextual degree of truth of the proposition. Consider for example the simple case of universal relation R. The membership value of Tom in uncertain set $\mathbf{Q_h}$ is a measure of the valuations that find Qa true. If this is 0.6 then there is nothing "fuzzy" in this set membership because the value has a very precise meaning: 6 out of 10 valuations of logic $\mathcal{L}$ find "Tom is tall" is true.

Contextual degrees of truth as discussed in section 5, can be generated from semantic relations R over H. Many predicates in ordinary language like "tall" are not precise and their truth-values depend on context, on truth-values assigned to other propositions as well. In the earlier discussion relations were defined among valuations according to the propositions they find true.

Relation F was defined among valuations assigning truth-values to propositions describing Tom's family for example, while B holds between valuations assigning truth-values to propositions about basketball. Since Tom's family are very short, but basketball players are tall, p = "Tom is tall" is true more often among the valuations related by F in H, than in general, and far more often than among valuations related by B. In our example the the *family-wise-degree-of-truth* of p was 0.9, $deg_F(p) = 0.9$, compared with the general degree of 0.6, deg(p) = 0.6, while the basketball-wise-degree-of-truth was much lower, $deg_B(p) = 0.3$.

These relations and contextual degrees generate corresponding contextual uncertain sets. Tom is a 0.6 member according to our definition of the general uncertain set of tall men, while he is a 0.9 member of the family-wise-uncertain set, of "family-wise-tall-men". His membership in the "basketball-wise-uncertain set of tall men" is much lower, at 0.3. Each set membership expresses a corresponding contextual degree of truth of proposition p in the global logic $\mathcal{L}$. Relation F generates from predicate Q the *family-wise uncertain set* $\mathbf{Q_F}$ whose membership function assigns the contextual uncertainties deg_F discussed above. Tom is a 0.9 member of this set because his family are short, and so 9 out of 10 valuations describing his male family find p true, though in general only 6 out of 10 valuations in H do so. His membership in $\mathbf{Q_B}$ the uncertain basket-ball tall men is much lower at 0.3 because only 3 out of 10 of these valuations describing men playing basketball find p true.

Since the uncertain sets developed here have many-valued membership functions which are the defining property of Zadeh's "fuzzy" sets, it seems global logic $\mathcal{L}$ may provide a foundation for these as well. The many-valued membership functions are degrees of truth, measures of the related valuations that find the corresponding predicate proposition true. The global connective uncertain inference seems a likely candidate for "fuzzy implication", since it was proved capable in earlier sections of expressing the conditional probabilities of classical and quantum theories. "Fuzzy logic" then, could be the many-valued global predicate logic $\mathcal{L}$ expressing degrees of truth.

The many-valued valuations $\mathbf{h}$ of this predicate logic can also be expressed in terms of corresponding crisp sets. Each valuation h in H defines a crisp set $\mathbf{Q}_h$ from predicate Q, where $a \in \mathbf{Q}_h$ iff h'(Qa) = **t**. It follows that the uncertain set $\mathbf{Q}_h$ can be also expressed as a measure of these crisp sets. For the membership function of this set has values that are degrees of truth, measures of the valuations related to h that find Qa true. This however is also a measure of the crisp sets $\mathbf{Q}_h$ containing a. The general membership function $\mathbf{X}_{Qh}$ of uncertain set $\mathbf{Q}_h$ tells us what many-valued $\mathbf{h}$ tells us, that 6 out of 10 crisp sets of tall men contain Tom.

Global logic $\mathcal{L}$ has been derived only from traditional primitives of logic, including two truth-values true and false as well as from truth-functional connectives that express Boolean Laws of Thought. These concepts alone allowed the derivation of the global connectives, including the many-valued degrees of truth and the uncertain fuzzy sets that can be derived from them. Thus the logical values in the interval [0, 1] are not primitive but measures of sets of propositions that are true, or of corresponding crisp sets. Fuzzy logic is rich and intuitive and is founded on first principals of logic.

The development of logic presented here can be briefly summarised. True and false are primitive concepts of logic, as are the traditional Boolean Laws of thought. From these are derived set H of valuations of truth-functional logic $\mathscr{L}$, the assignments of truth-values to propositions. However from thse truth-functional valuations a global logic of modalities is derived, that can express sophisticated uncertainties including classical and quantum probabilities as well as fuzzy sets. In this development only traditional primitives are allowed, and none is rejected.

Bivalent propositional calculus is now understood as the truth-functional logic of any context, of the truth-sets and falsity-sets of a valuation where all propositions are assigned a truth-value. The truth-functional $\mathscr{L}$ generalises the bivalent logic by allowing truth-value "gaps", essential if any sense of uncertainty can be expressed. This is the truth-functional logic of a valuation h in H. But richer than both of these truth-functional systems is the global logic based on the set H of valuations of $\mathscr{L}$. This logic can express far more than any Kripke logic and has a far more intuitive foundation in logical first principles. This logic can express ordinary language contextual uncertainties and degrees of truth, as well as classical and quantum probabilities and fuzzy sets.

Notes

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[i] This means for any **a, b** in A there are elements (**a** $\wedge$ **b**), (**a** $\vee$ **b**) in A such that the first is a

[ii] See Birkhoff Latticce Theory for more.

[iii] So named in honour of Tarski who developed it and Lindenbaum a fellow logician who was murdered by the Nazis

[iv] That is we need to show for example that if $\gamma \in [\alpha]$ and $[\alpha] \leq [\beta]$, then $[\gamma] \leq [\beta]$: this follows from the fact that $\gamma \equiv \alpha$ is logically true in $\mathscr{L}$ and so $\alpha \supset \beta$ is logically true iff $\gamma \supset \beta$ is logically true, and similarly for the other cases.

[v] The Boolean properties follow from the valuation rules defining connectives, e.g. because $(\alpha \vee \beta) \equiv (\beta \vee \alpha)$ is logically equivalent in logic $\mathscr{L}$, $[\alpha \vee \beta] = [\beta \vee \alpha]$ and so **a** $\vee$ **b** = **b** $\vee$ **a** ie. this operation commutes. Similarly other properties follow from those of logical connectives establishing that $\mathscr{A}$ = <A, $\wedge$, $\vee$, $\leq$ > is a distributive lattice

[vi] See Garden 1984 for details

[vii] These results follow from the fact that logically equivalent propositions share the same valuation relations, ie. they have the same values in any h.

[viii] See for example Sikorski [1969].

[ix] Most notably of course by Birkhoff and von Neumann who have argued that quantum logic is not distributive, see more below and in Garden [1984]

[x] I have elsewhere called these Kolmogorov measures since they are well-defined over a Boolean field of sets, however this has caused difficulties eg. For Streater and so this reference to Kolmogorov has been avoided here.
[xi] See Garden [1984] and [1992] for a detailed refutation of von Neuman's and Birkhoff's claim that distribution must fail: their argument proves only that either distribution or Boolean negation fails, and of course we expect the latter to do so.
[xii] See for example the discussion in Hajek and Zadeh and Kaprzyk ed.

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